Solutions to the Gradiometer Inverse Problem

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Although the gravity gradiometer inverse problem is non-unique, there is a diversity of relevant information which can be used to constrain solutions, and extract interesting information from the data. We propose a novel approach to the gradiometry inverse problem which incorporates the following innovations. First, the solutions sought are *perturbations about some baseline model*, in which the gradient tensor field is predicted uniquely in the *forward direction.* Secondly, the perturbations in the mass density to be inferred from the data are modelled in terms of three dimensional spatially localized functions in the Earth's interior. Finally, we expand the gravitational potential in terms of two dimensional spatially localized functions on the sphere, (the orbit of the spacecraft). The expansion of the gravitational potential on the sphere with localized basis functions other than the traditional spherical harmonic basis is a powerful approach to the problem because it is exactly *local fluctuations in the mass density* about the baseline model which we wish to detect.

In more detail, we propose to model the mass density interior to the Earth $\rho(\mathbf{r})$ as the linear combination

$$\rho(\mathbf{r}) = \rho_0(\mathbf{r}) + \delta\rho(\mathbf{r}) \tag{1}$$

where $\rho_0(\mathbf{r})$ is any assumed initial guess and $\delta\rho(\mathbf{r})$ are fluctuations about that guess. The solution of the gravitational potential from Laplace's equation can be written

$$\begin{aligned}
\Phi_0(\mathbf{r}) &= \int d\mathbf{r}'\ G(\mathbf{r},\mathbf{r}')\rho_0(\mathbf{r}) \\
\Phi(\mathbf{r}) &= \Phi_0(\mathbf{r}) + \int d\mathbf{r}'\ G(\mathbf{r},\mathbf{r}')\delta\rho(\mathbf{r})
\end{aligned} \tag{2}$$

where $G(\mathbf{r},\mathbf{r}')$ is the Green's function of Laplace's equation.

The first equation above is solved uniquely in the forward direction from the assumed $\rho_0(\mathbf{r})$. We already have code which can, for a specified initial mass density model, solve for the gravitational potential (mention the finite element code and collaborators at Boulder).

The fluctuations in mass density $\delta\rho(\mathbf{r})$ will be represented as a linear combination of localized functions. To provide some intuition about this approach, consider the paricularly simple example given by the initial model is a uniform mass density $\rho(\mathbf{r})$. Furthermore, if a hole is then excavated at some depth in the Earth's interior, the exact solution to the gravitational potential would simply be that given by a uniforma density sphere with a hole. The gradient tensor would only involve the term from the hole, and the only two parameters of the problem would be the size and location of the hole. Therefore, the simpler a solution is, the better constrained it is by the data.

This example is obviously not realistic. However, it is relevant to consider localized fluctuations in mass density about some local mean density, particularly when we are interested in detecting changes in the mass density, as the result of geophysical activity. Given a partial understanding of the relevant geophysics in various regions in the earth, the type of fluctuations in mass density can be constrained.

In particular, the optimal language to use in modeling the density fluctuations comes directly from the physics of the problem. By a "local density fluctuation" we essentially mean a deviation of the mass density from a local average. The physics of the problem suggests that we think of local averages according to

$$\text{Local Average} \propto \int d\mathbf{r}\; e^{-(\mathbf{r}-\mathbf{b})\cdot\mathbf{A}\cdot(\mathbf{r}-\mathbf{b})}\rho(\mathbf{r}) \tag{3}$$

where the matrix $\mathbf{A}$ describes an ellipsoidal shape characterized in the three principal orthogonal directions as $\mathbf{A} = \text{diag}[\lambda_1, \lambda_2, \lambda_3]$. A fluctuation is then given roughly by the difference of the mass density at some point $\rho(\mathbf{r})$ with the local average at the same point. More generally, we can characterize fluctuations of a certain scale according to the Laplacian of localized Gaussians

$$\text{Fluctuation} \propto \int d\mathbf{r}\; \nabla^2 e^{-(\mathbf{r}-\mathbf{b})\cdot\mathbf{A}\cdot(\mathbf{r}-\mathbf{b})}\rho(\mathbf{r}) \tag{4}$$

For example, a hole at some location in the Earth's interior $\mathbf{b}$ is given by a cirularly symmetric Gaussian with $\lambda_1 = \lambda_2 = \lambda_3$. Cylindrical fluctuations can be more accurately represented as a string of elongated Gaussian blobs.

Using the localized basis functions, we will assume the mass density fluctuations can be written as a linear superposition

$$\delta\rho(\mathbf{r}) = \sum_i F(\mathbf{b}_i) \left(\nabla^2 e^{-(\mathbf{r}-\mathbf{b}_i)\cdot\mathbf{A}\cdot(\mathbf{r}-\mathbf{b}_i)}\right) \tag{5}$$

It can be shown that, using the Laplacian of Gaussian blobs, that the resulting gravitational potential on the sphere is given directly as a linear combination of Gaussians, or

$$\Phi(\mathbf{r}) - \Phi_0(\mathbf{r}) = \sum_i F(\mathbf{b}_i) e^{-(\mathbf{r}-\mathbf{b}_i)\cdot\mathbf{A}\cdot(\mathbf{r}-\mathbf{b}_i)} \tag{6}$$

Simply taking the gradient tensor of this equation gives

$$\nabla_\alpha \nabla_\beta \delta\Phi(\mathbf{r}) = \sum_i F(\mathbf{b}_i) \left(\nabla_i \nabla_j e^{-(\mathbf{r}-\mathbf{b})\cdot\mathbf{A}\cdot(\mathbf{r}-\mathbf{b})}\right) \tag{7}$$

The important property of the functions

$$\left(\nabla_i \nabla_j e^{-(\mathbf{r}-\mathbf{b})\cdot\mathbf{A}\cdot(\mathbf{r}-\mathbf{b})}\right) \tag{8}$$

is that they provide the ability to characterize the fluctuations locally (to some chosen scale).

The above dealt with the tensor everywhere, but we will only observe the gradient tensor on a sphere, along the orbit of a spacecraft. An observed fluctuation in the gradient tensor is most strongly influenced by the region of the earth directly below the spacecraft. The ability to detect mass density fluctuations can be enhanced if the response of the gradient tensor to fluctuations below the spacecraft and close by is characterized.

Following this intuition, we can therefore choose some localized basis on the sphere of radius R (where the position in three dimensional space is now $\mathbf{r} = R\mathbf{n}$, and $\mathbf{n}$ a unit vector pointing to some point on the sphere). One example of localized functions on the sphere are to choose two dimensional Laplacian of "Gaussians". These functions need to be carefully defined on the sphere, but for the purposes of the present discussion we can approximately think of these functions as given by

$$g(\mathbf{n},\mathbf{n}';\sigma) \propto ||R\mathbf{n} - R\mathbf{n}'||^2 e^{-||R\mathbf{n}-R\mathbf{n}'||^2/2\sigma^2} \tag{9}$$

These collections of functions provide an alternate way to represent functions on the sphere than the spherical harmonics. Taking the "wavelet" transform on the sphere decomposes any function, such as the gravitational potential, into the coefficients

$$S(\mathbf{n},\sigma) = \int d\mathbf{n}' \; g(\mathbf{n},\mathbf{n}';\sigma)\Phi(R\mathbf{n}') \tag{10}$$

Although these functions are not orthogonal, there is amazingly still a reconstruction formula for the potential given by

$$\Phi(R\mathbf{n}) = \int \frac{d\sigma}{\sigma} \; d\mathbf{n}' \; S(\mathbf{n}',\sigma) \; g(\mathbf{n},\mathbf{n}';\sigma) \tag{11}$$

These coefficients are physically related to the mass density fluctuations according to

$$\begin{aligned} S(\mathbf{n},\sigma) &= \int d\mathbf{n}' \; g(\mathbf{n},\mathbf{n}';\sigma)\delta\Phi(R\mathbf{n}') \\ &= \int d\mathbf{n}' \; g(\mathbf{n},\mathbf{n}';\sigma)\left[\sum_i F(\mathbf{b}_i)e^{-(R\mathbf{n}'-\mathbf{b}_i)\cdot\mathbf{A}\cdot(R\mathbf{n}'-\mathbf{b}_i)}\right] \\ &= \sum_i F(\mathbf{b}_i)W[\mathbf{n};\sigma,\mathbf{b}_i] \end{aligned} \tag{12}$$

The kernel $W[\mathbf{n};\sigma,\mathbf{b}_i]$ is a very interesting quantity physically, and measures the sensitivity of a fluctuation in the gravitational potential on a sphere of radius R to a fluctuation in the interior of the Earth (at position $\mathbf{b}_i$), and given by

$$W[\mathbf{n};\sigma,\mathbf{b}_i] = \int d\mathbf{n}' \; g(\mathbf{n},\mathbf{n}';\sigma)e^{-(R\mathbf{n}'-\mathbf{b}_i)\cdot\mathbf{A}\cdot(R\mathbf{n}'-\mathbf{b}_i)} \tag{13}$$

The important property of response W is again that is much more localized than, for example, a spherical harmonic decomposition. Fluctuations in the gravitational potential at a point on the sphere are much more strongly influenced by fluctuations closest to the spacecraft (i.e. immediately below). Finally, the gravitational potential

$$\Phi(R\mathbf{n}) = \sum_i F(\mathbf{b}_i) \left[\int \frac{d\sigma}{\sigma}\, d\mathbf{n}'\; g(\mathbf{n}, \mathbf{n}'; \sigma) W[\mathbf{n}'; \sigma, \mathbf{b}_i] \right] \tag{14}$$

Taking the gradient tensor of the above then related the observations from the gradiometer to the mass density fluctuations coefficients $F(\mathbf{b}_i)$ which we want to infer from the data.

1. Goals

(Just to list them for now - I will re-write this) Two principal directions for effort - the first being progress in the forward modelling effort, and the second being in the computational developement of the wavelet approach. More specifically, we might propose to:

- Write code that computes fast wavelet transforms on the sphere.
- Study the response kernel with the goals of
  - Compute the response functions for various types of mass density fluctuations. This will quantify the acheivable level of resolution for the mass density fluctuations.
  - Compute the Singular Value Decomposition eigenvectors of the matrix W on a discrete grid of points on the Earth and the orbit. A basis in the null space will then characterize the variations in mass density which we are insensitive too. Are there other observations or cosntraints which can be employed?

- Attempt to characterize the ability to reconstruct the mass density in the presence of detailed baseline models. Attempt to include topolography maps of the Earth in order to define fluctuations about some baseline model of mass density.