Six-dimensional $\mathrm{N}=2$, $\mathrm{F}_4/\mathrm{SO}(9)$ Supergravity and the roof of the Magic Square

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Abstract

Motivated by the construction by Günaydin, Sierra and Townsend of Maxwell-Einstein $\mathrm{N}=2$ supergravity theories in 3, 4 and 5 dimensions realizing symmetries appearing in the lower three rows of the 4×4 Freudenthal-Tits Magic Square, we present a six-dimensional theory with $F_4/\mathrm{SO}(9)$ structure, corresponding to the maximal "octonionic" item in the top row of the square. The construction involves detailed interplay between the scalar coset structure and the spin-1 sector comprised of vectors, self-dual and anti-self-dual antisymmetric tensors.

1. Introduction

The celebrated Magic Square of Freudenthal and Tits

	$\mathbb{A}=\mathbb{O}$	$\mathbb{A}=\mathbb{H}$	$\mathbb{A}=\mathbb{C}$	$\mathbb{A}=\mathbb{R}$
$d=6$	F_4	C_3	A_2	A_1
$d=5$	E_6	A_5	$\mathrm{A}_2 \times \mathrm{A}_2$	A_2
$d=4$	E_7	D_6	A_5	C_3
$d=3$	E_8	E_7	E_6	F_4

	$\mathbb{A}=\mathbb{O}$	$\mathbb{A}=\mathbb{H}$	$\mathbb{A}=\mathbb{C}$	$\mathbb{A}=\mathbb{R}$
$d=6$	$\frac{\mathrm{F}_{4(-20)}}{\mathrm{SO}(9)}$	$\frac{\mathrm{USp}(4,2)}{\mathrm{USp}(4)\times\mathrm{SU}(2)}$	$\frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$	$\frac{\mathrm{SO}(2,1)}{\mathrm{SO}(2)}$
$d=5$	$\frac{\mathrm{E}_{6(-26)}}{\mathrm{F}_4}$	$\frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)}$	$\frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)}$	$\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)}$
$d=4$	$\frac{\mathrm{E}_{7(-25)}}{\mathrm{E}_6\times\mathrm{U}(1)}$	$\frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6)\times\mathrm{U}(1)}$	$\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{U}(1)}$	$\frac{\mathrm{USp}(6)}{\mathrm{SU}(3)\times\mathrm{U}(1)}$
$d=3$	$\frac{\mathrm{E}_{8(-24)}}{\mathrm{E}_7\times\mathrm{SU}(2)}$	$\frac{\mathrm{E}_{7(-5)}}{\mathrm{SO}(12)\times\mathrm{SU}(2)}$	$\frac{\mathrm{E}_{6(2)}}{\mathrm{SU}(6)\times\mathrm{SU}(2)}$	$\frac{\mathrm{F}_{4(4)}}{\mathrm{USp}(6)\times\mathrm{SU}(2)}$

$d=6$	$\underline{26}=\underline{1}^{(\mathrm{t}+)}+\underline{16}^{(\mathrm{t}-)}+\underline{9}^{(\mathrm{v})}$ $52-36=16^{(0)}$	$\underline{14}=\underline{1}^{(\mathrm{t}+)}+\underline{8}^{(\mathrm{t}-)}+\underline{5}^{(\mathrm{v})}$ $21-(10+3)=8^{(0)}$	$\underline{8}=\underline{1}^{(\mathrm{t}+)}+\underline{4}^{(\mathrm{t}-)}+\underline{3}^{(\mathrm{v})}$ $8-(3+1)=4^{(0)}$	$\underline{5}=\underline{1}^{(\mathrm{t}+)}+\underline{2}^{(\mathrm{t}-)}+\underline{2}^{(\mathrm{v})}$ $3-1=2^{(0)}$
$d=5$	$\underline{27}=\underline{1}^{(\mathrm{v})}+\underline{26}^{(\mathrm{v})}$ $78-52=26^{(0)}$	$\underline{15}=\underline{1}^{(\mathrm{v})}+\underline{14}^{(\mathrm{v})}$ $35-21=14^{(0)}$	$\underline{9}=\underline{1}^{(\mathrm{v})}+\underline{8}^{(\mathrm{v})}$ $16-8=8^{(0)}$	$\underline{6}=\underline{1}^{(\mathrm{v})}+\underline{5}^{(\mathrm{v})}$ $8-3=5^{(0)}$
$d=4$	$\underline{28}=\underline{1}^{(\mathrm{v})}+\underline{27}^{(\mathrm{v})}$ $133-(78+1)=54^{(0)}$	$\underline{16}=\underline{1}^{(\mathrm{v})}+\underline{15}^{(\mathrm{v})}$ $66-(35+1)=30^{(0)}$	$\underline{10}=\underline{1}^{(\mathrm{v})}+\underline{9}^{(\mathrm{v})}$ $35-(8+8+1)=18^{(0)}$	$\underline{7}=\underline{1}^{(\mathrm{v})}+\underline{6}^{(\mathrm{v})}$ $21-(8+1)=12^{(0)}$
$d=3$	$248-(133+3)=112^{(0)}$	$133-(66+3)=64^{(0)}$	$78-(35+3)=40^{(0)}$	$52-(21+3)=28^{(0)}$

2. $F_{4(-20)}$ description in an SO(9) basis

$$\begin{pmatrix} Z_0 & \underline{1} \\ Z_i & \underline{9} \\ Z_a & \underline{16} \end{pmatrix}$$

$$\begin{aligned} \mathcal{Q} &\equiv \eta^{IJ}\, Z_I\, Z_J \\ &= Z_0^2 \;+\; Z_i\, Z_i \;\mp\; Z_a\, Z_a \end{aligned} \tag{1}$$

$$\begin{aligned} \mathcal{C} &\equiv c^{IJK}\, Z_I\, Z_J\, Z_K \\ &= Z_0^3 \;-\; 3\, Z_0\, Z_i\, Z_i \;\mp\; \frac{3}{2}\, Z_0\, Z_a\, Z_a \;\pm\; \frac{3\sqrt{3}}{2}\left(\Gamma_i^{ab}\right) Z_i\, Z_a\, Z_b \end{aligned} \tag{2}$$

$$\begin{pmatrix} 0 & 0 & \sqrt{3}Y^b \\ 0 & X^{ij} & \left(\Gamma_i^{bc}\right) Y^c \\ \sqrt{3}Y^a & \left(\Gamma_j^{ac}\right) Y^c & \frac{1}{4}\left(\Gamma_{k\ell}^{ab}\right) X^{k\ell} \end{pmatrix} \tag{3}$$

2. The scalar sector

$$\begin{pmatrix} v^I{}_0 & v^I{}_i & v^I{}_a \end{pmatrix} \begin{pmatrix} u^0{}_J \\ u^i{}_J \\ u^a{}_J \end{pmatrix} \;=\; \left(\delta^I{}_J\right) \tag{4}$$

$$\begin{pmatrix} u^0{}_I \\ u^i{}_I \\ u^a{}_I \end{pmatrix} \begin{pmatrix} v^I{}_0 & v^I{}_j & v^I{}_b \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta^i{}_j & 0 \\ 0 & 0 & \delta^a{}_b \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} u^0{}_I \\ u^i{}_I \\ u^a{}_I \end{pmatrix} \begin{pmatrix} \partial_\mu v^I{}_0 & \partial_\mu v^I{}_j & \partial_\mu v^I{}_b \end{pmatrix} \;=\; \begin{pmatrix} 0 & 0 & \sqrt{3}R_\mu^b \\ 0 & Q_\mu^{ij} & \left(\Gamma_i^{bc}\right) R_\mu^c \\ \sqrt{3}\,R_\mu^a & \left(\Gamma_j^{ac}\right) R_\mu^c & \frac{1}{4}\left(\Gamma_{k\ell}^{ab}\right) Q_\mu^{k\ell} \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} u^0{}_I \\ u^i{}_I \\ u^a{}_I \end{pmatrix} \begin{pmatrix} \mathcal{D}_\mu v^I{}_0 & \mathcal{D}_\mu v^I{}_j & \mathcal{D}_\mu v^I{}_b \end{pmatrix} \;=\; \begin{pmatrix} 0 & 0 & \sqrt{3}R_\mu^b \\ 0 & 0 & \left(\Gamma_i^{bc}\right) R_\mu^c \\ \sqrt{3}\,R_\mu^a & \left(\Gamma_j^{ac}\right) R_\mu^c & 0 \end{pmatrix} \tag{7}$$

$$\mathcal{D}_\mu v^I{}_0 \;\equiv\; \partial_\mu v^I{}_0 \tag{8}$$

$$\mathcal{D}_\mu v^I{}_i \;\equiv\; \partial_\mu v^I{}_i \;-\; Q_\mu^{ij}\, v_j^I \tag{9}$$

$$\mathcal{D}_\mu v^I{}_a \;\equiv\; \partial_\mu v^I{}_a \;-\; \tfrac{1}{4}\, Q_\mu^{ij}\left(\Gamma_{ij}^{ab}\right) v_b^I \tag{10}$$

The SO(9) Fierz identities

$$\Gamma_i^{a(b}\,\Gamma_i^{cd)} \;=\; \delta^{a(b}\,\delta^{cd)} \tag{11}$$

$$3\delta^{a[c}\,\delta^{d]b} + \Gamma_i^{a[c}\,\Gamma_i^{d]b} \;=\; \tfrac{1}{4}\,\Gamma_{ij}^{ab}\,\Gamma_{ij}^{cd} \tag{12}$$

3. The spin-1 sector

$$F^I_{\mu\nu} = 2\,\partial_{[\mu}A^I_{\nu]} \tag{13}$$
$$G^I_{\mu\nu\rho} = 3\,\partial_{[\mu}B^I_{\nu\rho]} + 3\,c^I_{JK}\,F^J_{[\mu\nu}\,A^K_{\rho]} \tag{14}$$

$$\delta A^I_\mu = \partial_\mu\lambda^I \tag{15}$$
$$\delta B^I_{\mu\nu} = 2\,\partial_{[\mu}\Lambda^I_{\nu]} + 3\,c^I_{JK}\,F^J_{\mu\nu}\,\lambda^K \tag{16}$$

$$\partial_{[\mu}F^I_{\nu\rho]} = 0 \tag{17}$$
$$\partial_{[\mu}G^I_{\nu\rho\sigma]} = 3\,c^I_{JK}\,F^J_{[\mu\nu}\,F^K_{\rho\sigma]} \tag{18}$$

$$F^0_{\mu\nu} \equiv u^0{}_I\,F^I_{\mu\nu} \tag{19}$$
$$F^i_{\mu\nu} \equiv u^i{}_I\,F^I_{\mu\nu} \tag{20}$$
$$F^a_{\mu\nu} \equiv u^a{}_I\,F^I_{\mu\nu} \tag{21}$$

$$F^0_{\mu\nu} = 0 \tag{22}$$
$$F^a_{\mu\nu} = 0 \tag{23}$$

$$G^0_{\mu\nu\rho} \equiv u^0{}_I\,G^I_{\mu\nu\rho} \tag{24}$$
$$G^i_{\mu\nu\rho} \equiv u^i{}_I\,G^I_{\mu\nu\rho} \tag{25}$$
$$G^a_{\mu\nu\rho} \equiv u^a{}_I\,G^I_{\mu\nu\rho} \tag{26}$$

$$G^0_{\mu\nu\rho} = +\widetilde{G}^0_{\mu\nu\rho} \tag{27}$$
$$G^a_{\mu\nu\rho} = -\widetilde{G}^a_{\mu\nu\rho} \tag{28}$$

$$G^i_{\mu\nu\rho} = 0 \tag{29}$$

4. Supersymmetry transformations and field equations

$$\delta\,e_{\mu\alpha} = \overline{\psi}_\mu\gamma^\alpha\epsilon \tag{30}$$
$$\delta\,\psi_\mu = \nabla_\mu\epsilon + \frac{1}{4}\,G^0_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon \tag{31}$$
$$\delta\,A^I_\mu = v^I{}_i\,\overline{\lambda}^i\gamma_\mu\epsilon \tag{32}$$
$$\delta\,B^I_{\mu\nu} = v^I{}_0\,\overline{\psi}_{[\mu}\gamma_{\nu]}\epsilon - \frac{1}{2}\,v^I{}_a\,\overline{\chi}^a\,\gamma_{\mu\nu}\epsilon + 2c^I_{JK}\,A^J_{[\mu}\delta A^J_{\nu]} \tag{33}$$
$$\delta\,\lambda^i = -\frac{1}{4}\,F^i_{\mu\nu}\,\gamma^{\mu\nu}\epsilon \tag{34}$$
$$\delta\,\chi^a = P^a_\mu\,\gamma^\mu\epsilon + \frac{1}{12}\,G^a_{\mu\nu\rho}\,\gamma^{\mu\nu\rho}\epsilon \tag{35}$$

$$
\begin{aligned}
[\ \delta_{s.s.}(\epsilon_2)\,,\,\delta_{s.s.}(\epsilon_1)\ ] \quad &= \quad \delta_{g.c.}(\xi^\mu)\ +\ \delta_{l.l.}(\Sigma^{\alpha\beta}) & (36)\\
&\quad +\ \delta_{gauge(v)}(\lambda^I)\ +\ \delta_{gauge(t)}(\Lambda^I_\mu) & (37)\\
&\quad +\ \delta_{SO(9)}(L^{ij}) & (38)
\end{aligned}
$$

$$
\begin{aligned}
\xi^\mu &= \bar{\epsilon}_2\gamma^\mu\epsilon_1 & (39)\\
\Sigma^\mu &= \xi^\mu({\omega_\mu}^{\alpha\beta} - {G^0}_\mu{}^{\alpha\beta}) & (40)\\
\lambda^I &= -\xi^\mu\, A^I_\mu & (41)\\
\Lambda^I_\mu &= \ldots & (42)\\
L^{ij} &= -\xi^\mu\, Q^{ij}_\mu & (43)
\end{aligned}
$$

5. Discussion

A. Spacetime conventions

$$
\widetilde{X}_{\mu\nu\rho} \quad = \quad \frac{1}{6}{e_{\mu\nu\rho}}^{\sigma\tau\kappa}\,X_{\sigma\tau\kappa} \qquad (44)
$$

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